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Role of parametric noise in nonintegrable quantum dynamics

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The effect of parametric perturbations on the quantum dynamics of a system with nonintegrable classical limit and a mixed phase space is shown to depend crucially on the local invariant phase space structures sampled by the evolution. Using a probe coherent state four distinct scenarios are considered: (i) large regular regions surrounding primary fixed points; (ii) smaller regular regions corresponding to higher order orbits; (iii) scarred states on hyperbolic fixed points; (iv) chaotic regions. The result that scarring stabilizes quantum dynamics against fluctuations is discussed in the general context of instability in the quantum evolution as well as in relation to recent experiments on microwave ionization of Rydberg atoms.

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Quantum dynamics having a nonintegrable classical limit with a mixed phase space are not as well understood as their counterparts with a strictly chaotic limit. The complicated quantum behavior results from coexisting regular and chaotic regions in the classical phase space, and the effects of these local features have been probed by numerical simulations as well as in laboratory experiments.

In the absence of parametric fluctuations, there is a marked difference in the classical and the quantum behavior of bounded, mixed dynamics. Whereas the classical dynamics can show arbitrarily complex behavior in time, the quantum case is limited to being quasiperiodic. However, a "Landau scenario" has been proposed for the approach to the semiclassical limit whereby more and more frequencies appear in the quantum behavior thus resolving the complex classical evolution on longer time scales [1]. An essential requirement for this increase in number of frequencies is that, starting with simple initial conditions (e.g., coherent state), the excited eigenstates of the quantum dynamics should display increasing complexity on approaching the semiclassical limit. However, as shown by Peres [2], this increased complexity is accompanied by a heightened sensitivity of the dynamics to perturbations.

Unitarity ensures that two slightly different initial conditions evolving under the same quantum dynamics maintain a constant overlap. This is not, however, true when the same initial state evolves under two slightly different dynamics. Stated differently, the behavior of the overlap under different dynamics depends strongly on the phase space structures probed by the state, with limiting cases of small periodic oscillations in classically regular regions and strong decay followed by quasiperiodic fluctuations in chaotic regimes [2].

These considerations are made more relevant to experiment by studying a dynamics in few degrees of freedom subject to parametric fluctuations. The issue now is how these fluctuations affect the stability of the dynamics. For instance, it is known that quasiperiodic [3,4] or random [5,6] external driving and phase randomization by measurement [7] can strongly alter stability and spectral features. Added motivation comes from recent experiments which assess the influence of controlled parametric noise on the quantum evolution in a mixed phase space [8].

Here we consider the mixed phase-space dynamics of the piecewise linear standard map [9,10], and demonstrate that the effect of fluctuations on the quantum evolution depends crucially on the invariant structures in phase space sampled by the dynamics.

As is well known, the evolution of a state $|\psi(t)\rangle$ is described by the quantum map

$$|\psi(t+1)\rangle = U(t)|\psi(t)\rangle \tag{1}$$

where the time-dependent, unitary evolution operator over a single kick is

$$U(t) = e^{-ip^2/2\hbar} e^{-iK(t)V(q)/\hbar} , \tag{2}$$

with integer t. The kicking potential $V(q)$ is a piecewise parabolic approximation of $\cos(q)$ leading to a modified standard map with piecewise linear forcing (for details see [9,10]). The position and momentum of a particle are denoted by (q,p) and $\hbar = 2\pi/N$ with integer N. Note that statements on the sizes of regions in phase space are with respect to $\hbar$. Instead of the usually constant kick strength K, we choose $K(t)$ to take a value K_1 or K_2 with probability 1/2. It is this random process that constitutes the parametric fluctuation.

The effect of the fluctuations is assessed by the return probability $P_r(t)$ which is defined as the overlap of $|\psi(t)\rangle$ with the initial state $|\psi(0)\rangle$:

$$P_r(t) = |\langle\psi(0)|\psi(t)\rangle|^2 . \tag{3}$$

In all cases $|\psi(0)\rangle = |q,p\rangle$ is a coherent state placed at

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the coordinates q and p. Two distinct averages are considered with $\langle P_r(t)\rangle$ denoting a simple time average while $\langle\langle P_r(t)\rangle\rangle$ involves both a temporal and an ensemble average over different realizations of $K(t)$. The time average is useful as it is inversely related to the number of states (in some representation like the eigenrepresentation of the dynamics in the absence of noise) excited by the coherent state (see Ref. [9] for details). Larger values of the time average mean the composition of the initial state is more robust [2] and, thus, less susceptible to noise.

When $|q,p\rangle$ lies in a large regular region like that surrounding a primary elliptic fixed point or low-order periodic orbit, the evolution is expected to be less susceptible to fluctuations. That is, sufficiently small amplitude fluctuations should not destroy the stability of the orbit. Figure 1 shows $\langle P_r(t)\rangle$ over a large range of fluctuations when the initial state is in the regular region near the primary elliptic fixed point. The kick strength K is either $K_1 = 1$ or $K_2 = K_1 + dK$ with equal probability. Note that the initial coherent state is well within the elliptic region for K as well as $K + dK$. It is clear that even 5% parametric noise levels (dK/K) have only a negligible influence on the stability of the dynamics. The reason is that the eigenstates excited by the coherent state are essentially the same for the two values of K. In this sense the part of the Hilbert space related to the regular region is robust against small "physical" perturbations having a smooth classical limit. Different realizations of the noise lead to fluctuations which are small compared with the large value of the time average. Thus the general effect is a plateau, for small dK, of height $\approx 1/\sqrt{N}$ and roughly constant width.

Note that for the piecewise linear map the eigenstates in the elliptic region are well approximated by harmonic oscillator eigenstates [10]. Thus, in principle, $\langle P_r(t)\rangle$ can be estimated in the semiclassical limit for $dK = 0$ and for

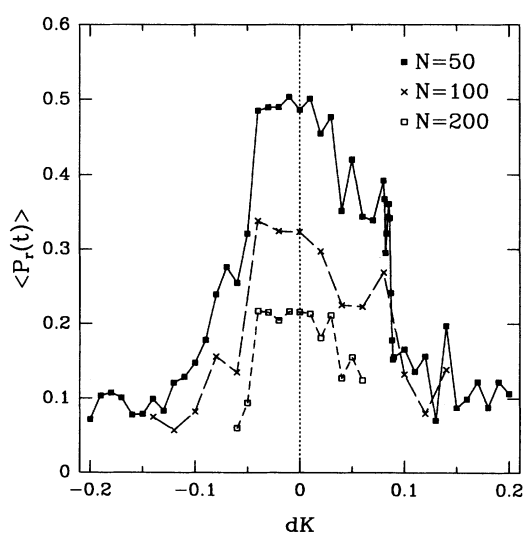

FIG. 1. The return probability $\langle P_r(t)\rangle$ averaged over the time interval $0 \le t \le 5000$, as defined in the text. The kick parameter varies between $K = 1$ and $1+dK$. The initial state is a coherent state placed at $q = 1.1\pi$ and $p = 0$ which is near the elliptic fixed point. Results are shown for three different values of $\hbar = 2\pi/N$ with $N = 50$, 100, and 200.

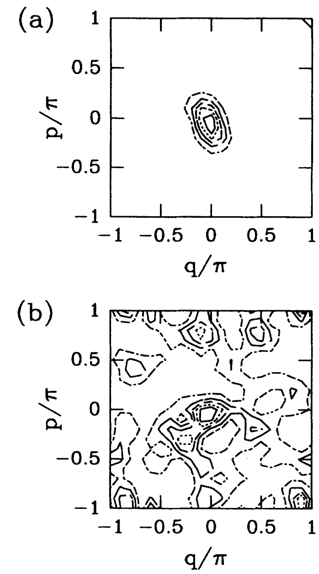

FIG. 2. The Husimi distribution $|\langle\psi(t)|q,p\rangle|^2$ for coherent initial state $|q_0,p_0\rangle = |1.1\pi, 0\rangle$ for $t = 3500$ using $K = 1$ and $dK = 0.08$ (a) and 0.09 (b).

a coherent initial state placed well inside the elliptic region. However, even in this simple case the result cannot be put in a closed form.

On increasing dK further the coherent state is destroyed in the long time limit, spreading over the allowed phase space. For $N = 50$ this happens rather abruptly for dK between 0.08 and 0.09. In Fig. 2 we compare the Husimi distributions obtained on iterating the same initial state inside the elliptic region for $t = 3500$ but with $dK = 0.08$ (a) and 0.09 (b). It is clear that the destruction of the coherent state is complete though the breakdown of coherence for $dK = 0.09$ occurs as early as $t = 1000$.

In Fig. 3, a coherent state was placed on an elliptic

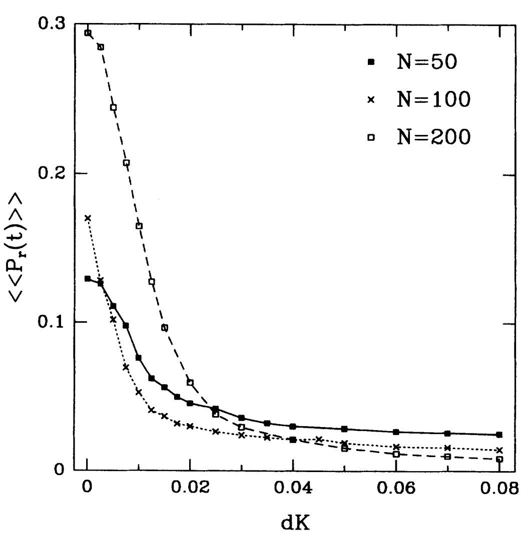

FIG. 3. The mean return probability $\langle\langle P_r(t)\rangle\rangle$ (averaged over the interval in Fig. 1) but for $K = 1.4$ and $1.4 + dK$. The initial state is a coherent state placed, near an elliptic 3-cycle, at $q = 0.23\pi$ and $p = 0.45\pi$. Three different values for N are used: 50, 100, and 200.

3-cycle of the piecewise linear standard map which is surrounded by small (of the order of $\hbar$) regular regions. Once again, the initial state covers the 3-cycle for both K and $K + dK$. The mean return probability shows a pronounced maximum for $dK = 0$ which decays strongly even for fluctuations of a few percent. Closer inspection of the dynamics reveals several processes. First, there is tunneling between the lobes of the 3-cycle which has only a moderate effect on $\langle\langle P_r(t)\rangle\rangle$. Next, tunneling exists between the 3-cycle and another symmetry related 3-cycle which can eventually lead to refocusing on the original 3-cycle. Lastly, for larger fluctuations there is the likelihood of spreading of the wave packet onto the neighboring heteroclinic tangles of the 3-cycle as well as the homoclinic tangle of the hyperbolic fixed point at the origin. This is the most likely cause for the very rapid decay of $P_r(t)$. It is worth noting that the lobes of the 3-cycle are of order $\hbar$ (for the N values chosen), which means they can each support one state. This leads to a simple estimate $1/6 < \langle P_r(t)\rangle < 1/3$ for small dK where the limiting values correspond to the excitation or not of the symmetry related 3-cycle.

The variation with $\hbar$ shows that smaller values provided greater containment of the initial state leading to enhanced return probabilities at lower noise levels. On the other hand, with increasing fluctuations and greater overlap with the tangles, the lower values of $\hbar$ mean greater instability and hence smaller return probabilities. For $N = 50$ the initial coherent state was too large to be contained in the elliptic regions in the presence of the perturbation. Decreasing $\hbar$ reduces the spread of the coherent initial state in phase space and leads to better containment in the elliptic regions even for larger fluctuations as seen from the simulations.

The most interesting case involves initial states in the vicinity of unstable fixed points. Here the phenomenon of scarring (see Heller [11] and [9,10] for the specific map) enhances the return probability. As the strength of scarring seems to decrease at least as fast as $\hbar$ [12] we show results only for $N = 50$. The larger value of $\hbar$ is also appropriate for the experimental discussion which follows. In Fig. 4, we take $K = 4$ which guarantees large scale chaos and contrast $\langle\langle P_r(t)\rangle\rangle$ for three different initial conditions: (i) sufficiently far away from the scar, (ii) on the scar, and (iii) on the scar but with additional measures taken to destroy antiunitary symmetries. These symmetries result in enhanced return probability on certain lines of symmetry and especially at the origin (see [13]).

In the absence of strong scarring, the average return probability is only slightly influenced by fluctuations. This is in sharp contrast to cases (ii) and (iii) where a pronounced maximum is seen for sufficiently small fluctuations. The only difference is that (ii) does not decay to the random matrix limit [as do (i) and (iii)] because of the accidental antiunitary symmetries. Away from the hyperbolic point, the initial coherent state spreads quickly over the homoclinic tangle of the hyperbolic fixed point and ultimately covers almost the entire phase space uniformly. This leads to a mean return probability of around $1/N = 0.02$. In the other two cases, for sufficiently small fluctuations (few percent), the initial state spreads but is also refocused on a time scale of the order of N leading to a strongly enhanced $\langle\langle P_r(t)\rangle\rangle$. Further increase in noise levels leads to breakdown of scarring.

These results clearly indicate that scarring persists for sufficiently small fluctuations. The noise level needed to reduce the unperturbed return probability by a factor of 2, say, is much smaller than for the elliptic fixed point (Fig. 1) but only slightly smaller than for the elliptic 3-cycle (Fig. 2). It is certainly large enough to be observable even with moderate resolution of the parameter K.

On decreasing $\hbar$ (or increasing N), we find the return probability in the absence of fluctuations is reduced but the scarring is still clearly visible. On the other hand, the susceptibility to fluctuations is increased and scarring is destroyed by much smaller fluctuations at smaller values of $\hbar$. Therefore we conclude that the importance of scarring on approaching the semiclassical limit is ultimately limited by fluctuations in the parameters of the system [12].

Qualitatively, the results presented here are generic to any quantum dynamics in a mixed phase space. We illustrate this feature by relating our findings to recent experiments on the effects of noise on ionization thresholds of excited hydrogen atoms.

Quantum stability associated with invariant classical structures including scarring is relevant to interpreting ionization curves for hydrogen atoms interacting with microwave radiation [14,15]. This stability is reflected in the higher field values needed to produce the same degree of ionization and appears as bumps in the frequency dependent ionization threshold. Each bump is identified by the scaled frequency $\Omega_0 = n_0^3\omega$ defined as the ratio of microwave frequency ω to the Kepler frequency of the initial state (principal quantum number n_0) and the locations correlate both with stable and unstable (leading

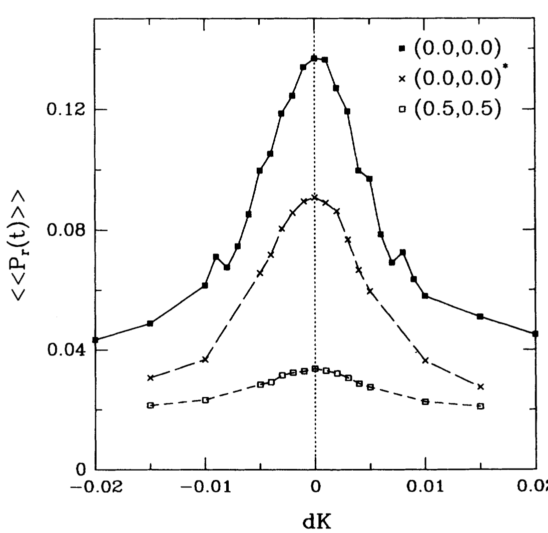

FIG. 4. The mean return probability $\langle\langle P_r(t)\rangle\rangle$ (as in Fig. 1) but for $K = 4$ and $4{+}dK$ and for $N = 50$. Two different initial coherent states are used: at the hyperbolic fixed point at the origin (with and without broken time reversal symmetry indicated by the asterisk) and at $(q,p) = (\pi/2, \pi/2)$.

to scarring) classical structures [15].

Recent experiments assess the effect of parametric noise on both types of stability [8]. Noise levels ranging from a few to 15% of the peak field values were considered. Stability associated with large regular regions as at $\Omega_0 \approx 1$ was seen to be much less susceptible to noise than when scarring was the cause, at $\Omega_0 \approx 1.3$. However, stability resulting from scarring survived for lower noise levels before rapidly decaying. At the highest noise levels, only the stability associated with the large regular region persists. These results are fully consistent with our findings.

Our results also indicate that careful preparation of the initial state sufficiently close to a hyperbolic fixed point is essential for the observability of the scar in experiment. This, too, is consistent with the analysis by Jensen *et al.* [14] of ionization experiments at $\Omega_0 \approx 1.3$ where the scarred state constituted 98% of the wave function at peak field. This should be kept in mind when assessing effects of noise on other experimentally observed scars.

The experiments also found instances of noise-induced stability, typically in frequency windows between the stable bumps. To illustrate this feature here we need to consider larger values of $\hbar$ corresponding to $N = 10-20$ [14], which are more consistent with the experiments. Figure 5 shows $\langle\langle P_r(t)\rangle\rangle$ for short times ($t \leq 100$) across a slice of phase space for $N = 10$, $K = 3$ and four noise levels. The two peaks at the ends correspond to scarring on the hyperbolic fixed point $q = 0$ and the stable region around the primary island at $q = \pi$. The resolution for $N = 10$ is too poor to see the effect of smaller structures in the classical phase space.

Noise-induced enhancement is clearly visible around $q = 0.7\pi$, in the vicinity of the large elliptic region. This enhancement is no longer seen for higher noise levels or at smaller values of $\hbar$. A possible explanation for the enhancement is that the fluctuations mix in the more stable eigenfunctions localized near or in the elliptic region

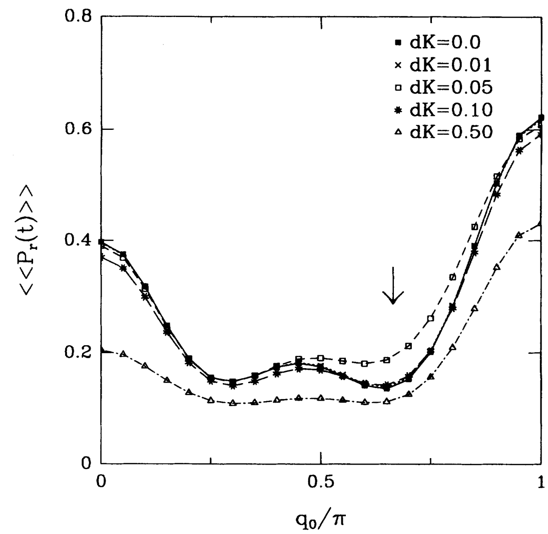

FIG. 5. $\langle\langle P_r(t)\rangle\rangle$ as a function of q for fixed $p = 0$. As indicated, four levels of noise are shown and $N = 10$. Note that the origin and $(q,p) = (\pi, 0)$ are the hyperbolic and elliptic fixed points, respectively. Noise-induced enhancement around $q = 0.7\pi$ is indicated by the arrow.

into the (otherwise unstable) eigenfunction composition of the initial state thereby increasing its return probability. Further increase of the fluctuations ultimately destabilizes all eigenstates leading to vanishing of the enhancement.

In conclusion, we have presented a simple analysis of the effects of parametric noise on quantum dynamics in a mixed phase space. We have also demonstrated that these features are generic by applying our findings to recent ionization experiments. Further, we speculate that the enhanced stability against small parametric fluctuations may be a useful signature for experimentally identifying the phenomenon of scarring.

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